V Money, Inflation and Monetary Policy (Continued)

2 The Sidrauski Model (Sidrauski, 1967)

The Sidraski model extends the Ramsey model to allow both consumption and real money balances to enter the utility function. The economy is populated by identical infinitely lived households, with population growing at a constant rate n.

2.1 Households

The representative household chooses per capita consumption c and per capita real money balances m to maximize its lifetime utility

$$V = \int_0^\infty e^{-\theta t} u(c, m) dt, \quad u_c, u_m > 0, u_{cc}, u_{mm} < 0, \tag{1}$$

subject to the following budget constraint[1]

$$\dot{k} + \dot{m} = w + rk - c - nk - (\pi + n)m + x, \tag{2}$$

where θ = the subjective discount rate, w = the real wage, r = the interest rate, k = per capita capital stock, π = the inflation rate and x = per capita government transfers. Defining total household wealth $A \equiv K + M/P$, then $a = k + m$. As a result, the household's budget constraint (2) becomes

$$\dot{a} = (r - n)a + w + x - c - (\pi + r)m. \tag{3}$$

The no-Ponzi-game condition is:

$$\lim_{t\to\infty} ae^{-\int_0^t (r_s - n)ds} = 0.$$

[1]This equation comes from $C + \dot{K} + \dot{M}/P = wN + rK + X$, where N, C, K, M, X and P are the household size, consumption, holdings of capital, nominal money, government transfers and the price level repectively.

We can solve the household's optimization problem by using the current-value Hamiltonian:

$$H = u(c,m) + \lambda[(r-n)a + w + x - c - (\pi + r)m], \tag{4}$$

where λ is the costate variable associated with (refbudget2). The first-order conditions are (4) and the following conditions:

$$\partial H/\partial c = u_c(c,m) - \lambda = 0, \tag{5}$$

$$\partial H/\partial m = u_m(c,m) - \lambda(\pi + r) = 0, \tag{6}$$

$$\partial H/\partial a = (r-n) = \theta\lambda - \dot{\lambda}, \tag{7}$$

$$\text{Transversality conditon:} \quad \lim_{t\to\infty} a\lambda e^{-\theta t} = 0 \tag{8}$$

From (5), we can see that λ is the marginal utility of consumption. From (5) and (6), we have

$$\frac{u_m(c,m)}{u_c(c,m)} = \pi + r. \tag{9}$$

If c and m are both normal goods (i.e., $u_{mm} - (u_m/u_c)u_{cm} < 0$ and $u_{cc} - (u_c/u_m)u_{cm} < 0$), then given c and $\pi + r$, (9) gives a unique m:

$$m = \tilde{m}(c, \pi + r)$$

with $\partial\tilde{m}/\partial c > 0$ and $\partial\tilde{m}/\partial(\pi + r) < 0$.

2.2 Firms

Firms do not need to use money. They use a CRS (constant returns to scale) technology and factor markets are competitive. As a result, the interest rate and the real wage rate are the respective marginal products of capital and labor:

$$r = f'(k), \tag{10}$$

$$w = f(k) - kf'(k). \tag{11}$$

2.3 Government

Assume that the money supply increases at a constant rate, i.e., $\dot{M}/M = \sigma$, and the money supply expands through lump-sum transfers to households, i.e., $\dot{M} = X$. This gives

$$x = \frac{\dot{M}}{PN} = \left(\frac{\dot{m}}{M}\right)\left(\frac{M}{PN}\right) = \sigma m. \tag{12}$$

Since $\dot{m}/m = \dot{M}/M - \dot{P}/P - \dot{N}/N$, then we have

$$\dot{m} = x - (\pi + n)m. \tag{13}$$

2.4 Steady-State Equilibrium

In the steady state, $\dot{a} = \dot{m} = \dot{\lambda} = 0$. First, (12) and (13) and $\dot{m} = 0$ give

$$\pi = \sigma - n. \tag{14}$$

Second, combining (7) and (10) yields

$$f'(k^*) = \theta + n. \tag{15}$$

Third, using (4), (11) and (15), we have

$$c^* = f(k^*) - nk^*. \tag{16}$$

Finally, from (5), (6), (11) and (15), we obtain

$$u_m(c^*, m^*) = (\theta + \sigma)u_c(c^*, m^*) \quad \Rightarrow \quad m^* = \tilde{m}(c^*, \sigma + \theta). \tag{17}$$

2.5 Results

The following results are obtained:

- *Superneutrality:* The steady state levels of capital stock and consumption are independent of money growth, so money is superneutral.
- Effect on real money balances: The level of real money balances depends on money growth:

  $$\frac{dm}{d\sigma} = \frac{u_c}{u_{mm} - (u_m/u_c)u_{cm}},$$

  which is negative if both c and m are normal goods.
- Optimal quantity of money: The money supply expands at a rate such that $\pi = -r$ or $\sigma = -\theta$ [to ensure that $u_m(c, m) = 0$, see (6)]. This is consistent with Friedman result that it is optimal to satiate individuals with money and that the rate of return on money should be the same as on capital.