数理经济学教学大纲

内容架构

1. 数学：概念、定理、证明，重在直觉，多用图形；
2. 举例：数学例子；
3. 应用：经济学实例；

资料来源：

杰里和瑞尼：《高级微观经济理论》，上海财大出版社和培生集团，2002 年。(JR)
马斯-科莱尔、温斯顿顿和格林：《微观经济学》，中国社会科学出版社，2001 年。(MWG)
巴罗和萨拉伊马丁：《经济增长》，中国社会科学出版社，2000 年。(BS)
迪克西特：《经济理论中的最优化方法》，上海三联和上海人民出版社，2006 年。(KD)
德雷泽：《宏观经济学中的政治经济学》，经济科学出版社，2003 年。(AD)
斯托基和卢卡斯：《经济动态的递归方法》，中国社会科学出版社，1999 年。(SL)

集合与映射(JR，pp.387-435)

凸集
开球、闭球
开集、闭集
分离超平面(MWG p.1346)
有界集
紧集

连续性
(柯西)连续性
D 中的开集
D 中的闭集
连续性及其逆象

数列
收敛的数列
有界的数列
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威尔斯拉斯定理
布劳威不动点定理

实值函数
递增的、严格递增与强递增函数
递减的、严格递减与强递减函数
水平集
上优集与下劣集

凹函数
严格凹函数
拟凹函数
严格拟凹函数
凸与拟凸函数
凸与严格凸函数

微积分(JR,pp.436-483)

链式运算法则
凹性与一和二阶导数
方向导数
梯度
海赛矩阵
杨格定理

齐次函数
齐次函数的偏导数
欧拉定理

隐函数定理(BS p.490)
泰勒定理(BS p.491)

静态最优化(JR, pp.436-483)

实值函数局部内点最优化的一阶必要条件
实值函数局部内点最优化的二阶必要条件
海赛矩阵负定与正定的充分条件
实值函数局部内点最优化的充分条件
(无约束的)局部与全局最优化
严格凹性/凸性与全局最优化的惟一性
惟一全局最优化的充分条件

约束最优化
拉格朗日方法

拉格朗日定理

加边海赛矩阵
两变量、一约束最优化问题中的一个局部最优化的充分条件
等式约束条件下的局部最优化的充分条件
不等式约束
受非负性条件约束的实值函数最优化的必要条件

非线性规则问题
库恩-塔克条件
受不等式条件约束的实值最优化的(库恩-塔克)必要条件

值函数
包络定理

微分方程(BS,pp.438-466)

微分方程
　　一阶线性微分方程
　　二阶线性微分方程
微分方程的稳定性
　　一阶线性微分方程的稳定性
　　线性微分方程系统的稳定性
　　非线性微分方程系统的稳定性

动态优化

最优控制原理(BS, pp.466-487; KD, pp.130-144)
　　典型问题
　　极大值原理
　　横截性条件
　　现值和当期值汉密尔顿函数
　　多个变量

动态规划导论(KD，pp.145-159；AD，pp.28-33)
　　贝尔曼方程

CHAPTER 1 SETS AND MAPPINGS

1. Convex Set
$S \subset \mathbb{R}^n$ is a convex set if for all $x^2 \in S$, we have
$$tX^1 + (1-t)x^2 \in S$$
for all $t \in [0,1]$.
Convex set: the Intersection of Convex Sets is Convex

2.Open and Closed ε –Balls
Open ε –Ball with center x^0 and radius $\varepsilon > 0$ is the subset of points in $\mathbb{R}^n$,
$$B_\varepsilon(x^0) = \{x \in \mathbb{R}^n | d(x^0, x) < \varepsilon\}$$
Closed ε –Ball with center x^0 and radius $\varepsilon > 0$ is the subset of points in $\mathbb{R}^n$,
$$B_\varepsilon^*(x^0) = \{x \in \mathbb{R}^n | d(x^0, x) \leq \varepsilon\}$$
Open set: $S \subset \mathbb{R}^n$ is an open set if, for all $x \in S$, there exists some $\in > 0$ such that $B_\varepsilon(x) \subset S$.
Close set: S is closed set if its complements S^C is an open set.

Open sets:
The empty set, $\emptyset$, is an open set.
The entire space, $\mathbb{R}^n$, is an open set.
The union of open sets is an open set.
The intersection of any finite number of open sets is an open set.

Every Open Set is a Colletion of Open Balls:
Let S be an open set. For every $x \in S$, choose some $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset S$. Then,
$$S = \cup_{x \in S} B_{\varepsilon_x}(x).$$

Closed Sets:
The empty set, $\emptyset$, is a closed set.
The entire space, $\mathbb{R}^n$, is a closed set.
The union of any finite collection of closed sets is a closed set.
The intersection of closed sets is a close set.

3.Separating Hyperplanes
Given $p \in \mathbb{R}^n$ with $p \neq 0$, and $c \in \mathbb{R}$, the hyperplane generated by p

and c is set $H_{p,c} = \{z \in \mathbb{R}^n | p \cdot z = c\}$. The sets $\{z \in \mathbb{R}^n | p \cdot z \geq c\}$ and $\{z \in \mathbb{R}^n | p \cdot z \leq c\}$ are called, respectively, the half-space above and the half-space below the hyperplane $H_{p,c}$.

Separating Hyperplane Theorem:

Suppose that $B \subset \mathbb{R}^n$ is convex and closed, and that $x \notin B$. Then there is $p \in \mathbb{R}^n$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot x > c$ and $p \cdot y < c$ for every $y \in B$.

More generally, suppose that the convex sets $A, B \subset \mathbb{R}^n$ are disjoint (i.e., $A \cap B = \emptyset$). Then there is $p \in \mathbb{R}^n$ with $p \neq 0$, and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in A$ and $p \cdot y \leq c$ for every $y \in B$. That is, there is a hyperplane that separates A and B, leaving A and B on different sides of it.

Supporting Hyperplane Theorem:

Suppose that $B \subset \mathbb{R}^n$ is convex and that x is not an element of the interior of set B (i.e., $x \notin \text{Int } B$). Then there is $p \in \mathbb{R}^n$ with $p \neq 0$ such that $p \cdot x \geq p \cdot y$ for every $y \in B$.

4.Bounded sets

Bounded sets: S is bounded if there exists some $\varepsilon > 0$ such that

$S \subset B_\varepsilon(x)$ for some $x \in \mathbb{R}^n$.

Lower bound: any real number l for which $l \leq x$ for all $x \in S$ is called a lower bound for S.

Upper bound: any real number u for which $x \leq u$ for all $x \in S$ is called an upper bound for S.

$S \subset \mathbb{R}$ is bounded from below if it has a lover bound, and is bounded from above if it has an upper bound.

The largest number among the lower bounds is called the greatest lower bound $(g.l.b.)$ of S.

The smallest number among the upper bounds is called the least upper bound $(l.u.b.)$ of S.

Upper and Lower Bounds in Subsets of Real Numbers:

1. Let S be a bounded open set in $\mathbb{R}$ and let a be the $g.l.b.$ of S and b be the $l.u.b.$ of S. Then $a \notin S$ and $b \notin S$.

2. Let S be a bounded closed set in $\mathbb{R}$ and let a be the $g.l.b.$ of S and S and b be the $l.u.b.$ of S. Then $a \in S$ and $b \in S$.

(Heine-Borel) Compact sets: a set S in $\mathbb{R}^n$ is called compact if it is closed and bounded.

5. (Cauchy) Continuity

Let D be a subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}^n$. The function f is continuous at the point $x^0 \in D$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$f(B_\delta(x^0) \cap D) \subset B_\varepsilon(f(x^0)).$$

If f is continuous at every point $x \in D$, then it is called a continuous function.

Open sets in D : Let D be a subset of $\mathbb{R}^n$. Then a subset S of D is open in D if for every $x \in S$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap D \subset S$.

Closed sets in D : Let D be a subset of $\mathbb{R}^m$. A subset S of D is closed in D if its complement in D is open in D.

Continuity and Inverse Images:

Let D be a subset of $\mathbb{R}^m$. The following conditions are equivalent:

1. $f : D \to \mathbb{R}^n$ is continuous.
2. For every open ball B in $\mathbb{R}^n$, $f^{-1}(B)$ is open in D.
3. For every open set S in $\mathbb{R}^n$, $f^{-1}(S)$ is open in D.

Theorem *The Continuous Image of a Compact Set is a Compact Set*

Let D be a subset of $\mathbb{R}^m$ and let $f : D \to \mathbb{R}^n$ be continuous function. If $S \subset D$ is compact in D, then its image $f(S) \subset \mathbb{R}^n$ is compact in $\mathbb{R}^n$.

6. Sequence

Sequences in $\mathbb{R}^n$: A sequence of $\mathbb{R}^n$ is function mapping some infinite subset I of positive integers into $\mathbb{R}^n$. We shall denote a sequence by $\{x^k\}_{k\in I}$, where $x^k \in \mathbb{R}^n$ for every $k \in I$.

Convergent sequences: The sequence $\{x^k\}_{k\in I}$ converges to $x \in \mathbb{R}^n$ if for every $\varepsilon > 0$, there is a $\overline{k}$ such that $x^k \in B_\varepsilon(x)$ for all $k \in I$ exceeding $\overline{k}$.

Bounded sequences: A sequence $\{x^k\}_{k\in I}$in $\mathbb{R}^n$ is bounded if for some $M \in \mathbb{R}, \|x^k\| \leq M$ for all $k \in I$.

Subsequences: $\{x^k\}_{k\in J}$is a subsequence of the sequence $\{x^k\}_{k\in I}$in $\mathbb{R}^n$, if J is an infinite subset of I.

Bounded sequences:

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Sequence, Sets, and Continuous Functions:

Let D be subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}^m$. Then

1. D is open if and only if for each $x \in D$, if $\{x^k\}_{k=1}^{\infty}$ converges to x, then for some $\overline{k}$, $x^k \in D$ for all $k \geq \overline{k}$.

2. D is closed if and only if for every $\{x^k\}_{k=1}^{\infty}$ of points in D converging to some $x \in \mathbb{R}^n$, it is also the case that $x \in D$.

3. f is continuous if and only if whenever $\{x^k\}_{k=1}^{\infty}$ in D converges to $x \in D$, $\{f(x^k)\}_{k=1}^{\infty}$ converges to $f(x)$.

7. Some Existing Theorems

(Weierstrass) Existence of Extreme Values:

Let $f : S \to \mathbb{R}$ be a continuous real-valued mapping where S is a nonempty compact subset of $\mathbb{R}^n$. Then these exists a vector $x^* \in S$ such that

$$f(\tilde{x}) \leq f(x) \leq f(x^*) \text{ for all } x \in S.$$

The Brouwer Fixed-Point Theorem:

Let $S \subset \mathbb{R}^n$ be a nonempty compact and convex set. Let $f : S \to S$ be continuous mapping. Then there exists at least one fixed point of f in S. That is, thers exists at least one $x^* \in S$ such that $x^* = f(x^*)$.

8. Real-Valued Functions

Real-valued function: $f : D \to R$ is a real-valued function fi D is any set and $R \subset \mathbb{R}$.

Increasing, Strictly Increasing and Strongly Increasing Functions:

Let $f : D \to \mathbb{R}$, where D is subset of $\mathbb{R}^n$.

f is increasing if $f(x^0) \geq f(x^1)$ whenever $x^0 \geq x^1$.

f is strictly increasing if $f(x^0) > f(x^1)$ whenever $x^0 >> x^1$.

f is strongly increasing if $f(x^0) > f(x^1)$ whenever $x^0 \neq x^1$ and $x^0 \geq x^1$.

Decreasing, Strictly Decreasing and Strongly Decreasing Functions:

Let $f : D \to \mathbb{R}$, where D is subset of $\mathbb{R}^n$.

f is decreasing if $f(x^0) \leq f(x^1)$ whenever $x^0 \geq x^1$.

f is strictly decreasing if $f(x^0) < f(x^1)$ whenever $x^0 >> x^1$.

f is strongly decreasing if $f(x^0) < f(x^1)$ whenever $x^0 \neq x^1$ and $x^0 \geq x^1$.

9.Related sets

Level sets: $L(y^0)$ is a level set of the real-valued function $f : D \to R$ iff $L(y^0) = \{x | x \in D, f(x) = y^0\}$, where $y^0 \in R \subset \mathbb{R}$.

Level sets relative to a point: $\mathcal{L}(x^0)$ is a level set relative to x^0 if $\mathcal{L}(x^0) = \{x | x \in D, f(x) = f(x^0)\}$.

Superior set: $S(y^0) \equiv \{x | x \in D, f(x) \geq y^0\}$ is called the superior set for level y^0.

Inferior set: $I(y^0) \equiv \{x | x \in D, f(x) \leq y^0\}$ is called the inferior set for level y^0.

Strictly superior set: $S'(y^0) \equiv \{x | x \in D, f(x)' > y^0\}$ is called the strictly superior set for level y^0.

Strictly inferior set: $I'(y^0) \equiv \{x | x \in D, f(x) < y^0\}$ is called the strictly inferior set for level y^0.

Superior, Inferior, and Level Sets:

For any $f : D \to R$ and $y^0 \in R$:

1. $L(y^0) \subset S(y^0)$.
2. $L(y^0) \subset I(y^0)$.
3. $L(y^0) = S(y^0) \cap I(y^0)$.
4. $S'(y^0) \subset S(y^0)$.
5. $I'(y^0) \cap L(y^0) = \emptyset$.
6. $S'(y^0) \cap L(y^0) = \emptyset$.
7. $I'(y^0) \cap L(y^0) = \emptyset$.
8. $S'(y^0) \cap I'(y^0) = \emptyset$.

10. Concave Functions

Assumption:

(1) $D \subset \mathbb{R}^n$ is a convex set.

(2) When $x^1 \in D$ and $x^2 \in D$, $x^t = tx^1 + (1 - t)x^2$, for $t \in [0, 1]$, denote the convex combination of x^1 and x^2.

Concave function: $f : D \to R$ is a concave function if for all $x^1, x^2 \in D$,

$$f(x^t) \geq tf(x^1) + (1 - t)f(x^2) \ \forall t \in [0, 1]$$

Theorem *Points on and below the graph of a concave function is always form a convex set*

Let $A \equiv \{(\mathbf{x}, y) | \mathbf{x} \in D, f(\mathbf{x}) \geq y\}$ be the set of points "on and below" the graph of $f : D \to R$, where $D \subset \mathbb{R}$, Then,

f is a concave function $\Leftrightarrow$ A is a convex set.

Strictly concave function: $f : D \to R$ is a strictly concave function iff, for all $\mathbf{x}^1 \neq \mathbf{x}^2$ in D,

$$f(x^t) > tf(x^1) + (1 - t)f(x^2) \text{ for all } t \in (0, 1)$$

Quasiconcave function: $f : D \to R$ is quasiconcave iff, for all x^1 and x^2 in D,

$$f(x^t) \geq \min[f(x^1), f(x^2)] \text{ for all } t \in [0, 1]$$

Quasiconcavity and the superior sets

$f : D \to R$ is a quasiconcave function iff $S(y)$ is a convex set for all $y \in \mathbb{R}$.

Strictly quasiconcave function: A function $f : D \to R$ is strictly quasiconcave iff, for all $x^1 \neq x^2$ in D, $f(x^t) > \min[f(x^1), f(x^2)]$ for all $t \in (0, 1)$.

Concavity implies quasiconcavity:

A concave function is always quasiconcave.

A strictly concave functions is always strictly quasiconcave.

11. Convex and quasiconvex functions

Convex function: $f : D \to R$ is a convex function iff, for all x^1, x^2 in D,

$$f(x^t) \leq tf(x^1) + (1 - t)f(x^2) \text{ for all } t \in [0, 1].$$

Strictly convex function: $f : D \to R$ is a strictly convex function iff, for all $x^1 \neq x^2$ in D,

$$f(x^t) < tf(x^1) + (1 - t)f(x^2) \text{ for all } t \in (0, 1).$$

Concave and convex functions:

$f(x)$ is a (strictly) concave function $\Leftrightarrow$ $-f(x)$ is a (strictly) convex function.

Points on and above the graph of a convex function always form a convex set:

Let $A^* \equiv \{(\mathbf{x}, y)|\mathbf{x} \in D, f(\mathbf{x}) \leq y\}$ be the set of points "on and above" the graph of $f : D \to R$, where $D \subset \mathbb{R}^n$ is a convex set and $R \subset \mathbb{R}$. Then

$$f \text{ is a convex function } \Leftrightarrow A^* \text{ is a convex set.}$$

Quasiconvex function: $f : D \to R$ is quasiconvex$\Leftrightarrow$ $\forall x^1, x^2 \in D, f(x^t) \leq \max[f(x^1, f(x^2)]$

Strictly quasiconvex function: $f : D \to R$ is strictly quasiconvex$\Leftrightarrow$ $\forall x^1, x^2 \in D, f(x^t) < \max[f(x^1), f(x^2)]$

Quasiconvexity and the inferior sets:

$f : D \to R$ is quasiconvex function $\Leftrightarrow$ $I(y)$ is a convex set for all $y \in \mathbb{R}$.

Quasiconcave and quasiconvex functions:

$f(x)$ is a (strictly) quasiconcave function $\Leftrightarrow$ $-f(x)$ is a (strictly) quasiconvex function.

Remark

f is concave $\Leftrightarrow$ the set of points beneath the graph is convex
f is convex $\Leftrightarrow$ the set of points above the graph is convex
f quasiconcave $\Leftrightarrow$ superior sets are convex sets
f quasiconvex $\Leftrightarrow$inferior sets are convex sets
f concave $\Rightarrow$ f quasiconcave
f convex $\Rightarrow$ f quasiconvex
f (strictly) concave $\Leftrightarrow$ $-f$ (strictly) convex
f (strictly) quasiconcave $\Leftrightarrow$ $-f$ (strictly) quasiconvex

CHAPTER 2 CALCULUS

Functions of a Single Variable

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Concavity and first and second derivatives:

Let D be a nondegenerate interval of real numbers on which f is twice continuously differentiable. The following statements 1 to 3 are equivalent:

1. f is concave.
2. $f''(x) \leq 0 \ \forall x \in D$.
3. $\forall x^0 \in D : f(x) \leq f(x^0) + f'(x^0)(x - x^0) \ \forall x \in D$.

Moreover,

4. $f''(x) < 0 \ \forall x \in D \Rightarrow f$ is strictly concave.

Functions of Several Variables

Directional derivative:

$$\forall \mathbf{x}, \mathbf{z} \in \mathbb{R}, g(t) = f(\mathbf{x} + t\mathbf{z}) \Rightarrow g'(0) = \sum_{i=1}^{n} f_i(\mathbf{x})\mathbf{z}_i$$

The term on the right-hand side is known as the directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{z}$.

Gradient: $\nabla f(\mathbf{x}) \equiv (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is called the gradient of f at $\mathbf{x}$.

Remark $g'(0) = \nabla f(\mathbf{x})\mathbf{z}$.

Second-order partial derivative:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f(\mathbf{x})}{\partial x_1}\right) \equiv \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_1} \equiv f_{1i}(\mathbf{x})$$

Second-order gradient vector: the gradient of the partial with respect to x_1, $f_1(\mathbf{x})$.

$$\nabla f_1(\mathbf{x}) = \left(\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1}\right) \equiv (f_{11}(\mathbf{x}), \ldots, f_{1n}(\mathbf{x}))$$

Hessian matrix:

$$H(\mathbf{x}) = \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \ldots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \ldots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \ldots & f_{nn}(\mathbf{x}) \end{pmatrix}$$

Young's Theorem:

For any twice continuously differentiable function $f(\mathbf{x})$,

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i} \ \forall i \text{ and } j.$$

Single-Variable and Multivariable Concavity:

f is a real-value function defined on the convex subset D of $\mathbb{R}^n$. f is (strictly) concave $\Leftrightarrow$ $\forall \mathbf{x} \in D, \forall$nonzero $\mathbf{z} \in \mathbb{R}^n, g(t) = f(\mathbf{x} + t\mathbf{z})$ is (strictly) concave on $\{t \in \mathbb{R} | \mathbf{x} + t\mathbf{z} \in D\}$

Negative semidefinite matrix:

A is negative semidefinite $\Leftrightarrow$ $\forall z \in \mathbb{R}^n, z^T A z \leq 0.$

A is negative definite $\Leftrightarrow$ $\forall z \in \mathbb{R}^n, z^T A z < 0.$

A is positive semidefinite $\Leftrightarrow$ $\forall z \in \mathbb{R}^n, z^T A z \geq 0.$

A is positive definite $\Leftrightarrow$ $\forall z \in \mathbb{R}^n, z^T A z > 0.$

Slope,curvature, and concavity in many variables

Let D be a convex subset of $\mathbb{R}^n$ with a nonempty interior on which f is twice continuously differentiable. The following statements 1 to 3 are equivalent:

1. f is concave.
2. $H(x)$ is negative semidefinite for all x in D.
3. For $\forall x^0 \in D : f(x) \leq f(x^0) + \nabla f(x^0)(x - x^0) \ \forall x \in D.$

Moreover,

4. If $H(x)$ is negative definite for all x in D, then f is strictly concave.

Concavity, convexity, and second-order own partial derivatives

Let $f : D \to R$ be a twice diffentiable function.

1. f is concave $\Rightarrow$ $\forall x, f_{ii}(\mathbf{x}) \leq 0, i = 1, \ldots, n.$
2. f is convex $\Rightarrow$ $\forall x, f_{ii}(\mathbf{x}) \geq 0, i = 1, \ldots, n.$

Homogeneous Functions

Homogeneous function: A real-valued function $f(x)$ is homogeneous of degree k $\Leftrightarrow$ $f(tx) \equiv t^k f(x), \forall t > 0.$

$f(\mathbf{x})$ is homogeneous of degree 1, or linear homogeneous$\Leftrightarrow$ $f(tx) \equiv tf(x), \forall t > 0.$

$f(\mathbf{x})$ is homogeneous of degree zero$\Leftrightarrow$ $f(tx) \equiv f(x), \forall t > 0.$

Partial derivatives of homogeneous functions

If $f(\mathbf{x})$ is homogeneous of degree k, its partial deriatives are homogeneous of degree $k - 1$.

Euler's theorem (or adding-up theorem)

$f(\mathbf{x})$ is homogeneous of degree $k \iff \forall \mathbf{x}, kf(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i$

Some Useful Results in Calculus (BS)

Implicit function theorem

Taylor's theorem

CHAPTER 3 STATIC OPTIMIZATION

Unconstrained Optimization

Necessary conditions for local interior optima in the single-variable case

Let $f(x)$ be a twice continuously differentiable function of one variable. Then $f(x)$ reaches a local interior

1. maximum at x^* $\Rightarrow$ $f'(x^*) = 0$ (FONC),
   $\Rightarrow$ $f''(x^*) \leq 0$ (SONC).
2. minimum at $\tilde{x}$ $\Rightarrow$ $f'(\tilde{x}) = 0$ (FONC)
   $\Rightarrow$ $f''(\tilde{x}) \geq 0$ (SONC).

FONC for local interior optima of real-valued functions:

$f(\mathbf{x})$ reaches a local interior maximum or minimum at $\mathbf{x}^*$ $\Rightarrow$ $\nabla f(\mathbf{x}^*) = 0$

SONC for local interior optima of real-valued functions:

Let $f(x)$ be a twice continuously differentiable.

$f(x)$ reaches a local interior maximum at x^* $\Rightarrow$ $H(x^*)$ is negative semidefinite

$f(x)$ reaches a local interior maximum at $\tilde{x}$ $\Rightarrow$ $H(\tilde{x})$ is positive semidefinite

Sufficient condition for negative and positive definiteness of the Hessian:

Let $f(x)$ be twice continuously differentiable, and let $D_i(x)$ be the ith-order principal minor of the Hessian matrix $H(x)$.

1. $(-1)^i D_i(x) > 0, i = 1, \ldots, n$ $\Rightarrow$ $H(x)$ is negative definite.
2. $D_i(x) > 0, i = 1, \ldots, n$ $\Rightarrow$ $H(x)$ is positive definite.

If condition 1 holds for all x in the domain, then f is strictly concave.

If condition 2 holds for all x in the domain, then f is strictly convex.

Sufficient conditions for local interior optima of real-valued functions

Let $f(x)$ be twice continuously differentiable.

1. $f_i(x^*) = 0$ and $(-1)^n D_i(x^*) > 0, i = 1, \ldots, n$ $\Rightarrow$ $f(x)$ reaches a local maximum at x^*.
2. $f_i(\tilde{x}) = 0$ and $D_i(x^*) > 0, i = 1, \ldots, n$ $\Rightarrow$ $f(x)$ reaches a local minimum at $\tilde{x}$.

(Unconstrained) local-global theorem:

Let f be a twice continuously differentiable real-valued concave function on D. The following statements are equivalent, where x^* is an interior point of D :

1. $\nabla f(x^*) = 0$.

2. f achieves a local maximum at x^*.
3. f achieves a global maximum at x^*.

Stict concavity/convexity and the uniqueness of global optima:
1. x^* maximizes the strictly concave function $f \Rightarrow x^*$ is the unique global maximizer, i.e., $f(x^*) > f(x)\ \forall x \in D, x \neq x^*$.
2. $\tilde{x}$ minimizes the strictly convex function $f \Rightarrow \tilde{x}$ is the unique global minimizer, i.e., $f(\tilde{x}) < f(x)\ \forall x \in D, x \neq \tilde{x}$.

Sufficient condition for unique global optima:
Let $f(x)$ be twice continuously differentiable.
1. $f(x)$ is strictly concave and $f_i(x^*) = 0, i = 1, \ldots, n \Rightarrow x^*$ is the unique global maximizer of $f(x)$.
2. $f(x)$ is strictly convex and $f_i(\tilde{x}) = 0, i = 1, \ldots, n \Rightarrow \tilde{x}$ is the unique global maximizer of $f(x)$.

Constrained Optimization

Equality constraints

Two-variable, one constraint optimization problem:

$$\max_{x_1, x_2} f(x_1, x_2) \quad s.t. \quad g(x_1, x_2) = 0$$

Lagrange's method

1.Lagarangian function:

$$\mathcal{L}(x_1, x_2, \lambda) \equiv f(x_1, x_2) + \lambda g(x_1, x_2)$$

2.F.O.C

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \lambda^* \frac{\partial g(x_1^*, x_2^*)}{\partial x_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} + \lambda^* \frac{\partial g(x_1^*, x_2^*)}{\partial x_2} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0.$$

The first-order partials fo the Lagrangian function with respect to the x_i were

$$\mathcal{L}_i = f_i + \lambda g_i.$$

The second-order partials of $\mathcal{L}$ would then be

$$\mathcal{L}_{11} = f_{11} + \lambda g_{11}$$
$$\mathcal{L}_{12} = f_{12} + \lambda g_{12}$$
$$\mathcal{L}_{22} = f_{22} + \lambda g_{22}.$$

Bordered Hessian of the Lagrangian function

$$\overline{H} \equiv \begin{pmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$$

Sufficient condition for a local optimum:

$\overline{D} \equiv |\overline{H}| > 0 (< 0) \Rightarrow (x_1^*, x_2^*)$ is a local maximum (minimum)

n-variable, m-constraint optimization problem:

Lagrange's theorem:

Let $f(x)$ and $g^j(x), j = 1, \ldots, m,$ be continuously differentiable real-valued functions over some domain $D \subset \mathbb{R}^n$. Let x^* be an interior point of D and suppose that x^* is an optimum of f subject to the constraints, $g^j(x^*) = 0$. If $\nabla g^j(x^*)$ are linearly independent, then there exist m unique numbers λ_j^*, such that

$$\frac{\partial \mathcal{L}(x^*, \wedge^*)}{\partial x_i} = \frac{\partial f(x^*)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j^* \frac{\partial g^j(x^*)}{\partial x_i} = 0$$

Bordered Hessian matrix

$$\overline{H} = \begin{pmatrix} 0 & \cdots & 0 & g_1^1 & \ldots & g_n^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & g_1^m & \ldots & g_n^m \\ g_1^1 & \ldots & g_1^m & \mathcal{L}_{11} & \ldots & \mathcal{L}_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & \ldots & g_n^m & \mathcal{L}_{n1} & \ldots & \mathcal{L}_{nn} \end{pmatrix}$$

Sufficient conditions for local optima with equality constraints:

Let the objective function be $f(x)$ and the $m < n$ constraints be $g^j(x) = 0, j = 1, \ldots, m.$

1. x^* is a local maximum of $f(x)$ s.t. the constraints if the $n - m$ principal minors in $\overline{D} = |\overline{H}|$ alternate in sign beginning with positive

$\overline{D}_{m+1} > 0, \overline{D}_{m+2} < 0, \ldots,$ when evaluated at $(x^*, \wedge^*)$.

2. x^* is a local minimum of $f(x)$ s.t. the constraints if the $n - m$ principal minors in $\overline{D} = |\overline{H}|$ are all negative $\overline{D}_{m+1} < 0, \overline{D}_{m+2} < 0, \ldots,$ when evaluated at $(x^*, \wedge^*)$.

Necessary conditions for optima of real-valued functions subject to nonnegativity constraints

Let $f(x)$ be continuously differentiable.

1. If $\mathbf{x}^*$ maximizes $f(\mathbf{x})$ subject to $\mathbf{x} \geq 0$, then $\mathbf{x}^*$ satisfies

i. $\dfrac{\partial f(\mathbf{x}^*)}{\partial x_i} \leq 0, i = 1, \ldots, n$

ii. $x_i^*[\dfrac{\partial f(\mathbf{x}^*)}{\partial x_i}] = 0, i = 1, \ldots, n$

iii. $x_i^* \geq 0, i = 1, \ldots, n.$

2. If $\mathbf{x}^*$ minimizes $f(\mathbf{x})$ subject to $\mathbf{x} \geq 0$, then $\mathbf{x}^*$ satisfies

i. $\dfrac{\partial f(\mathbf{x}^*)}{\partial x_i} \geq 0, i = 1, \ldots, n$

ii. $x_i^*[\dfrac{\partial f(\mathbf{x}^*)}{\partial x_i}] = 0, i = 1, \ldots, n$

iii. $x_i^* \geq 0, i = 1, \ldots, n.$

Inequality constraints

A two-variable nonlinear programming programming

$$\max_{x_1, x_2} f(x_1, x_2)$$

$$s.t.\ g(x_1, x_2) \geq 0.$$

Equivalent to the three-variable problem with equality and nonnegativity constraints:

$$\max_{x_1, x_2, z} f(x_1, x_2)$$

$$s.t.\ g(x_1, x_2) - z = 0$$

$$z \geq 0.$$

Lagrangian function:

$$\mathcal{L}(x_1, x_2, z, \lambda) \equiv f(x_1, x_2) + \lambda[g(x_1, x_2) - z]$$

Kuhn-Tucker conditions:

$$\mathcal{L}_1 = f_1 + \lambda g_1 = 0$$

$$\mathcal{L}_2 = f_2 + \lambda g_2 = 0$$

$$\lambda g(x_1, x_2) = 0$$

$$\lambda \geq 0, g(x_1, x_2) \geq 0$$

(Kuhn-Tucker) Necessary conditions for optima of real-valued functions subject to inequality constraints

Let $f(\mathbf{x})$ and $g^j(\mathbf{x}), j = 1, \ldots, m,$ be continuously diffentiable real-valued functions over some domain $D \subset \mathbb{R}^n$. Let $\mathbf{x}^*$ be an interior point of D and suppose that $\mathbf{x}^*$ is an optimum of f subject to the constraints, $g^j(\mathbf{x}^*) \geq 0, j = 1, \ldots, m.$

If $\nabla g^j(\mathbf{x}^*)$ associated with all binding constraints are linearly independent, then there exists a unique vector $\wedge^*$such that $(\mathbf{x}^*, \wedge^*)$ satisfies the Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \wedge^*)}{\partial x_i} \equiv \frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j^* \frac{\partial g^j(\mathbf{x}^*)}{\partial x_i} = 0, i = 1, \ldots, n$$

$$\lambda_j^* g^j(\mathbf{x}^*) = 0 \quad g^j(\mathbf{x}^*) \geq 0 \quad j = 1, \ldots, m.$$

Furthermore, the vector $\wedge^*$ is nonnegative if $\mathbf{x}^*$ is a maximum, and nonpostitive if it is a minimum.

Value Functions

Envelop Theorem:

Consider the optimization problem (P1):

$$\max_x f(\mathbf{x}, a)$$

$$\text{s.t. } g(\mathbf{x}, a) = 0$$

$$x \geq 0.$$

suppose the objective function and constraint are continuously differentiable in $a,$ let $x(a) >> 0$ uniquely solve P1 and assume that it is also continuously differentiable in the parameters a. Let $\mathcal{L}(x, a, \lambda)$ be the problem's associated Lagrangian function and let $(x(a), \lambda(a))$ solve the Kuhn-Tucker conditions. Finally, let $M(a)$ be the problem's associated maximum-value function. Then,

$$\frac{\partial M(\mathbf{a})}{\partial a_j} = \left.\frac{\partial \mathcal{L}}{\partial a_j}\right|_{x(\mathbf{a}), \lambda(\mathbf{a})} \quad j = 1, \ldots, m.$$

where the right-hand side denotes the partial derivative of the Lagrangian function with respect to the parameter a_j evaluated at the point $(x(\mathbf{a}), \lambda(\mathbf{a}))$.

CHAPTER 4 Differential Equations

INTRODUCTION

A *differential equation* is an equation that involves derivatives of variables.

If there is only one independent variablce, then it is called an *ordinary differential equation (ODE)*.

If the highest derivative is an ODE is of order n, then it is an nth-order ODE.

When the functional form of the equation is linear, then it is a *linear ODE*.

Example: A first-order linear ODE

$$a_1 \cdot \dot{y}(t) + a_2 \cdot y(t) + x(t) = 0$$

where $\dot{y}(t) \equiv \dfrac{dy(t)}{dt}$

$x(t)$: forcing function

If $x(t) = a_3$, then the equation is called autonomous.

If $x(t) = 0$, then the equation is called homogeneous.

Solution methods

1.Graphical

- Used for nonlinear, as well as linear, differential equations;
- Used only for autonomous equations.

2.Analytical

- Used only with a limited set of functional forms.

3.Numerical analysis

- e.g. **Matlab** has the subroutines *ODE23* and *ODE45*, and **Mathematica** has the command *NDSOLVE*.

First-order ODE

Graphical solutions

- **1**. Consider an automoumous ODE of the form,

$$\dot{y}(t) = f[y(t)]$$

- Example 1

$$\dot{y}(t) = f[y(t)] = a \cdot y(t) - x$$

- Case 1, $a > 0$

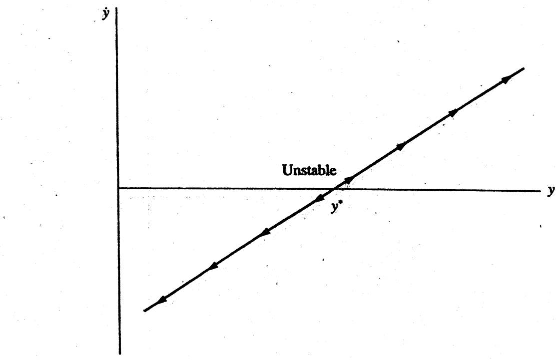

FIGURE A.1a
Linear ODE. If the coefficient a in Eq. (A.6) is positive, then the differential equation for y is unstable.

- Case 2, $a < 0$

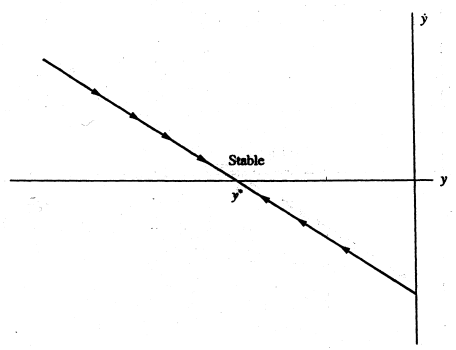

FIGURE A.1b
Linear ODE. If the coefficient a in Eq. (A.6) is negative, then the differential equation for y is stable.

- - Example 2

$$\dot{y}(t) = f[y(t)] = s \cdot [y(t)]^{\alpha} - \delta \cdot y(t)$$
$$s, \delta, \alpha > 0; \alpha < 1$$

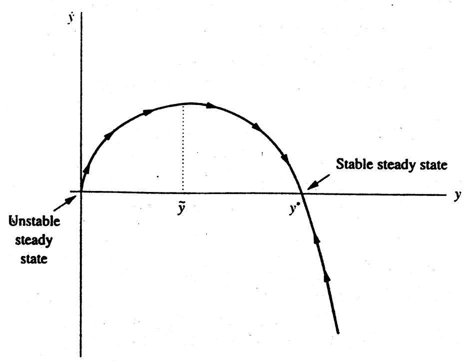

FIGURE A.1c
Nonlinear ODE. In Eq. (A.7), the slope of $f(\cdot)$ with respect to y is initially positive and is subsequently negative. The steady state at 0 is unstable, whereas that at y^* is stable.

Definition: ***Stability***

- - If $\left.\frac{\partial \dot{y}}{\partial y}\right|_{y^*} > 0$, then y is locally *unstable*,
  - If $\left.\frac{\partial \dot{y}}{\partial y}\right|_{y^*} < 0$, then y is locally *stable*.

Analytical Solutions

- The solution to $\dot{y}(t) = a$ is obviously $y(t) = b + at$, where b is an arbitrary constant.

- Equations that involve polynomial functions of time

$$\dot{y}(t) = a_0 + a_1 t + a_2 \cdot t^2 + ... + a_n \cdot t^n$$

  - has the solution

$$y(t) = b + a_0 t + a_1 \cdot \left(\frac{t^2}{2}\right) + ... + a_n \cdot \left(\frac{t^{n+1}}{n+1}\right)$$

- The general solution for linear, first-order ODEs
  - Linear, first-order differential equations with constant coefficients.

$$\dot{y}(t) + a \cdot y(t) + x(t) = 0$$

$$(1) \Rightarrow \dot{y}(t) + a \cdot y(t) = -x(t)$$

$$(2) \Rightarrow \int e^{at}[\dot{y}(t) + a \cdot y(t)]dt = -\int e^{at} \cdot x(t)dt$$

The term e^{at} is called the ***integrating factor***. The reason for multiplying by the integrating factor is that the term inside the left-hand side integral becomes the deriative of $e^{at} \cdot y(t)$ with respect to time:

$$e^{at} \cdot [\dot{y}(t) + a \cdot y(t)] = \frac{d}{dt}[e^{at} \cdot y(t) + b_0]$$

- - Hence, the term on the left-hand side of Eq.(2) equals $e^{at} \cdot y(t) + b_0$.

  (3) Compute the integral on the right-hand side of Eq.(2).

  Call the result $INT(t) + b_1$

  (4) Mutiply both sides by e^{-at} to get $y(t)$:

  $y(t) = -e^{-at} \cdot INT(t) + be^{-at}$

  where $b = b_1 - b_0$ is an arbitrary constant.

Problem Sets

- Ex.1 Show the general solution to $\dot{y}(t) - y(t) - 1 = 0$
  [Answer: $y(t) = -1 + be^{t}$]
- Ex.2 Suppose $\dot{k}(t) = \lambda[k(t) - k^*]$ and $k(0)$ is the initial value of $k(t)$, show that $k(t) = k^* + e^{-\lambda t}[k(0) - k^*]$. (Romer, 2001, p.24)

Linear, first-order differential equations with variable coefficients.

$$\dot{y}(t) + a(t) \cdot y(t) + x(t) = 0$$

- The integrating factor is now $e^{\int_0^t a(\tau)d\tau}$, so that the left-hand side becomes the derivative of $y(t) \cdot e^{\int_0^t a(\tau)d\tau}$.
- We find that the solution to the ODE is

$$y(t) = -e^{-\int_0^t a(\tau)d\tau} \cdot \int e^{\int_0^t a(\tau)d\tau} \cdot x(t) \cdot dt + b \cdot e^{-\int_0^t a(\tau)d\tau},$$

  - Where b is an arbitrary constant of integration.

Systems of Linear ODE

A system of linear, first-order ODEs of the form

$$\dot{y}_1(t) = a_{11} \cdot y_1(t) + \ldots + a_{1n} \cdot y_n(t) + x_1(t),$$
$$\ldots$$
$$\dot{y}_n(t) = a_{n1}y_1(t) + \ldots + a_{nn}y_n(t) + x_n(t).$$

In matrix notation, the system is

$$\dot{y}(t) = A \cdot y(t) + x(t),$$

Where $y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$

$$\dot{y}(t) = \begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{pmatrix}$$

A is an $n \times n$ square matrix of constant coefficients

$x(t)$ is a vector of n functions.

Solutions Methods

- Phase diagram
- Analytical
- Numerical

Phase Diagram

- Diagonal systems: A simple case

$$\dot{y}_1(t) = a_{11} \cdot y_1(t),$$
$$\dot{y}_2(t) = a_{22} \cdot y_2(t),$$

- where a_{11} and a_{22} are real numbers.

Case 1, $a_{11} > 0$ and $a_{22} > 0$: An unstable system

- - ***Step One***

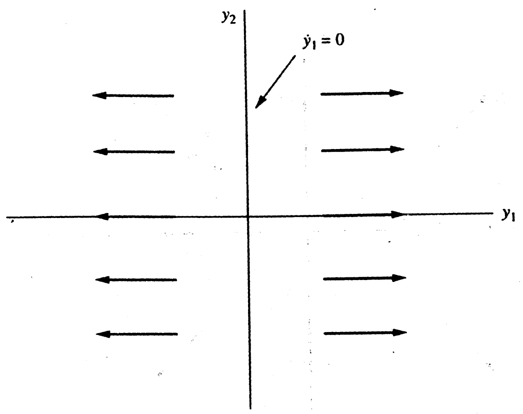

FIGURE A.4a
The $\dot{y}_1 = 0$ locus. The figure shows the $\dot{y}_1 = 0$ schedule (the vertical axis in this example) for the system in Eq. (A.20) when $a_{11} > 0$. The arrows show the direction of motion for y_1.

- ***Step Two***

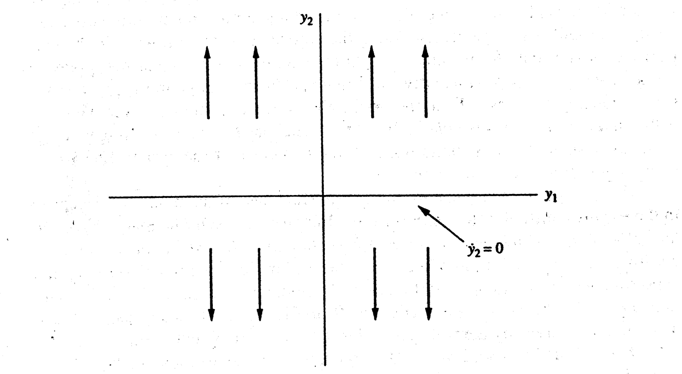

FIGURE A.4b
The $\dot{y}_2 = 0$ locus. The figure shows the $\dot{y}_2 = 0$ schedule (the horizontal axis in this example) for the system in Eq. (A.20) when $a_{22} > 0$. The arrows show the direction of motion for y_2.

- ***Step Three***

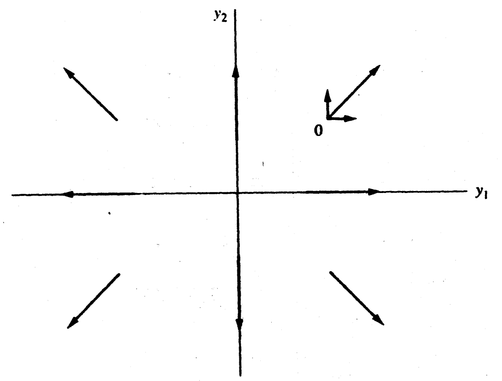

FIGURE A.4c
The phase diagram in an unstable case. The results from Figs. A.4a and A.4b are joined to generate a simple phase diagram. The arrows show the directions of motion for y_1 and y_2 when $a_{11} > 0$ and $a_{22} > 0$. This system is unstable.

- ***Step Four***, use the boundary conditions to see which one of the many paths depicted in the picture constitutes the exact solution.

Case 2, $a_{11} < 0$ and $a_{22} < 0$: A stable system

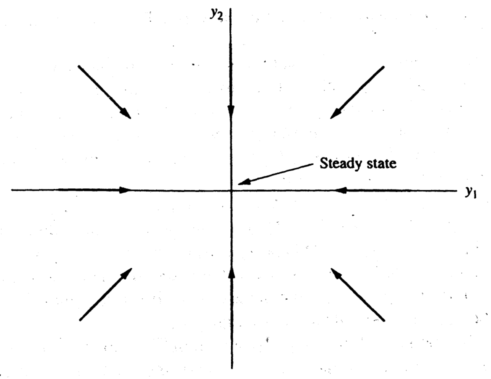

FIGURE A.5
The phase diagram in a stable case. In this example, $a_{11} < 0$ and $a_{22} < 0$ apply in Eq. (A.20). This system is stable.

Case 3, $a_{11} < 0$ and $a_{22} > 0$: A saddle-path stability.

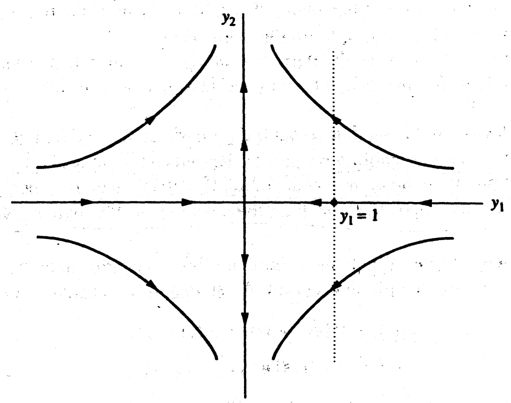

FIGURE A.6
The phase diagram in a case of saddle-path stability. In this example, $a_{11} < 0$ and $a_{22} > 0$ apply in Eq. (A.20). This system is saddle-path stable.

A nondiagonal example [see BS, p.476)]\

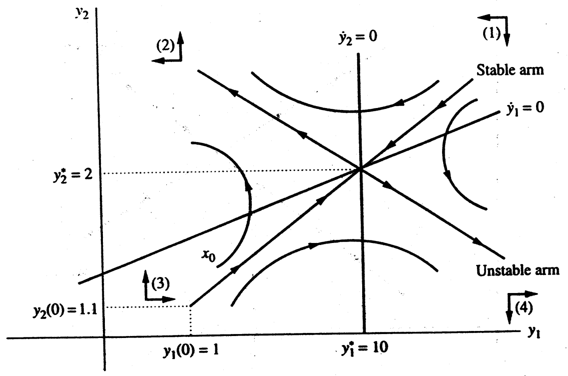

FIGURE A.7a
The phase diagram in a nonlinear example with saddle-path stability. The figure shows the phase diagram for the system in Eq. (A.21). This system is saddle-path stable.

CHAPTER 5 Dynamic Optimization

Dynamic Control

Typical Problem

$$\max_{c(t)} V(0) = \int_0^T v[k(t), c(t), t]dt$$

$$s.t.\ \dot{k}(t) = g[k(t), c(t), t] \qquad (1)$$

$$k(0) = k_0 > 0 \qquad (2)$$

$$k(T) \cdot e^{-r(T) \cdot T} \geqslant 0 \qquad (3)$$

where $V(0)$ –objective function as seen from the initial moment;

$r(t)$ –average discount rate

T –terminal planning date, finite or infinite

$k(t)$ –state variable

$c(t)$ –control variable

$v(\bullet)$ –instantaneous felicity functions

Eq.(1): transition equation *or* equation of motion

Eq.(2): the initial condition

Eq.(3): the final constraint

- For finite values of T, this constraint implies $k(T) \geqslant 0$, as long as the discount rate $r(T)$ is positive and finite. If $k(t)$ represents a person's net assets and T is the person's lifetime, then the constraint in Eq.(3) precludes dying in debt.
- If the planning horizon is infinite, then the condition says that net assets can be negative and grow forever in magnitude, as long as the rate of growth is less than $r(t)$. This constraint rules out *chain letters* or *Ponzi schemes for debt*.
- Example

$$v(k, c, t) = e^{-\rho t} \cdot u[c(t)]$$

$$\dot{k} = g[k(t), c(t), t] = f[k(t), t] - c(t) - \delta \cdot k(t)$$

Procedure to Find the First-Order Conditions

1. Hamiltonian function:

$$H = v(k, c, t) + \mu(t) \cdot g(k, c, t).$$

2. Take the derivative of the Hamiltonian with respect to the control variable and set it to 0:

$$\frac{\partial H}{\partial c} = \frac{\partial v}{\partial c} + \mu \cdot \frac{\partial g}{\partial c} = 0.$$

3. Take the derivative of the Hamiltonian with respect to the state variable and set it to equal the negative of the derivative of the multiplier with respect to time:

$$\frac{\partial H}{\partial k} \equiv \frac{\partial v}{\partial k} + \mu \cdot \frac{\partial g}{\partial k} = -\dot{\mu}.$$

4. *Step four (transversality condition)*:

- Case 1: Finite horizons. Set the product of the shadow price and the capital stock at the end of the planning horizon to 0:

$$\mu(T) \cdot k(T) = 0$$

- Case 2: Infinite horizons with discounting.

$$\lim_{t \to \infty}[\mu(t) \cdot k(t)] = 0.$$

- Case 3: Infinite horizons without discounting. In this case, we use Michel's condition,

$$\lim_{t \to \infty}[H(t)] = 0$$

Present-Value and Current-Value Hamiltonians

- Most of the models that we deal with have an objective function of the form,

$$\int_0^T v[k(t), c(t), t] \cdot dt = \int_0^T e^{-\rho t} \cdot u[k(t), c(t)] \cdot dt$$

- Constructiong the Hamiltonian

$$H = e^{-\rho t} \cdot u(k, c) + \mu \cdot g(k, c, t).$$

The shadow price $\mu(t)$ represents the value of the capital stock at time t in units of time-zero utils.

- Rewrite the Hamiltonian as

$$H = e^{-\rho t} \cdot [u(k, c) + q(t) \cdot g(k, c, t)],$$

where $q(t) \equiv \mu(t) \cdot e^{\rho t}$. The variable $q(t)$ is the current-value shadow price.

- Define $\hat{H} = H \cdot e^{\rho t}$ to be the current-value Hamiltonian:

$$\hat{H} \equiv u(k, c) + q(t) \cdot g(k, c, t).$$

- The first-order conditions can be expressed as

$$(4) \quad \hat{H}_c = 0,$$

$$(5) \quad \hat{H}_k = \rho q - \dot{q}.$$

- The transversality condition can be expressed as

$$q(T) \cdot e^{-\rho T} \cdot k(T) = 0.$$

Multiple Variables

- A general dynamic problem:

$$
\begin{aligned}
&\max_{c_1(t),\ldots,c_n(t)} \int_0^T u[k_1(t),\ldots,k_m(t);c_1(t),\ldots,c_n(t);t] \cdot dt \\
s.t.\ \dot{k}_1(t) &= g^1[k_1(t),\ldots,k_m(1);c_1(t),\ldots,c_n(t);t] \\
\dot{k}_2(t) &= g^2[k_1(t),\ldots,k_m(1);c_1(t),\ldots,c_n(t);t] \\
&\ldots \\
\dot{k}_m(t) &= g^m[k_1(t),\ldots,k_m(1);c_1(t),\ldots,c_n(t);t] \\
k_1(0) &> 0,\ldots,k_m(0) > 0, \textit{given} \\
k_1(T) &\geq 0,\ldots,k_m(T) \geq 0, \textit{free}
\end{aligned}
$$

- The Hamiltonian is

$$H = u[k_1(t),\ldots,k_m(t);c_1(t),\ldots,c_n(t);t] + \sum_{i=1}^{m} \mu_i \cdot g^i(\bullet).$$

- FONC

$$
\begin{aligned}
\frac{\partial H}{\partial c_i(t)} &= 0, i = 1,\ldots,n, \\
\frac{\partial H}{\partial k_i(t)} &= -\dot{\mu}_i, i = 1,\ldots,m,
\end{aligned}
$$

The transversality conditions are

$$\mu_i(t) \cdot k_i(T) = 0, i = 1,\ldots,m.$$

Dynamic Programming: An Introduction

Example: An optimal growth problem

$$
\begin{aligned}
\max_{c_t,k_t} U(t) &= \sum_{t=0}^{\infty} \beta_t u(c_t) \\
s.t.\ c_t + k_{t+1} &= f(k_t)
\end{aligned}
$$

Where $u(t)$ –instantaneos utility, which is an increasing, concave function of current consumption

β_t –individual's subjective discount factor, $\beta_t \in (0,1)$

c_t –current consumption

k_{t+1} –capital to be carried over to the following period

$$c_t + k_{t+1} = f(k_t)$$

where c_t – current consumption

k_{t+1} – capital to be carried over to the following period

Value function

- The value function represents the fact that the maximum present discounted value of the objective function from a point in time onward can be expressed as a function of the state variables at that date.

- Since the state variables at a point in time fully determine all other variables both currently and (via the transition equations) at all future dates, including those which enter the objective function, they determine the maximum attainable value of the objective function.
- Example:

$$\begin{tikzcd}
k_t \arrow[r] & k_{t+1} \arrow[r] \arrow[d] & k_{t+2} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
 & c_t \arrow[d] & c_{t+1} \arrow[d] & \cdots \arrow[d] \\
 & u(c_t) & u(c_{t+1}) & \cdots
\end{tikzcd}$$

  - The state variable k_t determines k_{t+1}, which determines k_{t+2}, et cetera.
  - k_t therefore determines the utility-maximizing value of c_t, c_{t+1}, c_{t+2}, et cetera, and this maximum attainable value is simply $V(k_t)$.
  - Therefore

$$V(k_t) = \max_{c_t, k_t} U(t) = \sum_{t=0}^{\infty} \beta_t u(c_t)$$

Bellman equation

1. In any period t, where the planner begins with k_t, the choice between c_t and k_{t+1} can be represented as maximizing

$$u(c_t) + \beta V(k_{t+1})$$

2. Combining two observations, we have

$$V(k_t) = \max_{k_{t+1}} \{u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})\} \quad \text{(eq.4)}$$

This is the Bellman equation, whose solution is a function $V(\bullet)$.

What lies behind the Bellman equation?

1. The value function allows a nontrivial dynamic problem to be turned into a simple-looking single-period optimization.

2. Dynamic problems are two-period problems balancing "today" against "the infinite future", but this works only when there is consistency between how one treated the future yesterday and how one treats it today.

Methods of solution

1. Method of conjecture.
2. Method of successive approximations.

The most useful method via the Bellman equation is not to solve for the value function $V(\bullet)$ at all, but to derive the optimal path without finding $V(\bullet)$ itself. Differentiability of $V(\bullet)$ allows us to do this in many cases.

With smooth functions and interior solutions, marginal changes in the state variable k imply marginal changes in attainable welfare in the same direction.

With differentiability of $V(\bullet)$, the maximum problem in (eq.4) yields the following first-order condition for k_{t+1} :

$$u'(f(k_t) - k_{t+1}) = \beta V'(k_{t+1}) \quad \text{(eq.5)}$$

Straightforward interpretation: choose k_{t+1} so that the loss in utility from "one less unit" of consumption today is just equal to the (discounted) gain in future "one more unit" of capital carried over (i.e. "saving") would allow.

What is the utility gain from higher k_{t+1}?

This can be calculated by use of the envelope theorem applied to $V(\bullet)$ to find the derivative $V'(\bullet)$, namely,

$$u'(f(k_t) - k_{t+1}) = u'(f(k_t) - k_{t+1})f'(k_t) \quad \text{(eq.6)}$$

Straightforward interpretation: the value of another unit of beginning-of-period capital along an optimal path is the marginal utility value of the extra product the higher capital allows evaluated at the optimal level of consumption.

Combining (eq.5) and (eq.6), one obtains

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - \kappa(k_{t+1}))f'(k_{t+1}) \quad \text{(eq.7)}$$

where $\kappa(k_{t+1}) \equiv k_{t+2}$.

Hence the time-invariant function $\kappa(\bullet)$ giving the optimal level of the control variable as a funtion of the state, sometimes called the policy function, which characterize the optimal path.

For many functional forms of $u(\bullet)$ and $f(\bullet)$, it is easy to solve (eq.7) to obtain a closed form for $\kappa(\bullet)$.

More generally, we can analyze this equation to obtain certain characteristics of $\kappa(\bullet)$ and hence the optimal path.